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Harmonic Univalent Functions Associated with ℓ −Hypergeometric Functions

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ABSTRACT: In the present work, we consider the function representing a rapidly convergent power series which extends the well-known confluent hypergeometric function $_1F_1$ [z] as well as the integral function $f(z)=\sum_{n=1}^{\infty}\frac{z^n}{n!^n}$. We study certain harmonic univalent mappings involving the ℓ-Hypergeometric functions. We establish the characteristics connected with harmonic mappings and mention their sufficient conditions.

Keywords: Analytic function, Univalent function, Convolution operator, Harmonic function, ℓ-Hypergeometric function, Simply-connected domain, Unit disk.

I. INTRODUCTION

Let $\mathbb{C}$ be a complex plan and $\mathfrak{D}$ be a simply connected domain on it. Let u and v be real-valued harmonic functions in $\mathfrak{D}$, then we call $f=u+iv$ is a complex valued harmonic function in $\mathfrak{D}$. If $f=h+\overline{g}$, then $|g'(z)|<|h'(z)|$ is the sufficient condition for f to be locally univalent and sense-preserving in $\mathfrak{D}$. Let $\mathcal{A}(\mathbb{D}_1(0))$ be the class of analytic functions in the open unit disk $\mathbb{D}_1(0)=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathcal{C}$ be the class of all functions $f\in\mathcal{A}(\mathbb{D}_1(0))$ which are normalized by $f(0)=0$ and $f'(0)=1$ and have the form

$$f(z)=z+\sum_{n=2}^{\infty}\alpha_n z^n,\quad z\in\mathbb{D}_1(0) \tag{1}$$

In 1984, Clunie and Sheil-small [1] introduced a class $\mathcal{SH}$ of complex-valued harmonic maps f which are univalent and sense-preserving in $\mathcal{C}$. So, for $f=h+\overline{g}$, it can be expressed in the analytic functions h and g as

$$h(z)=z+\sum_{n=2}^{\infty}\alpha_n z^n,$$
$$g(z)=\sum_{n=1}^{\infty}\beta_n z^n,|\beta_1|<1 \tag{2}$$

Let $\mathcal{S}$ be the class of normalized analytic univalent functions. Then $\mathcal{SH}$ reduces to the class $\mathcal{S}$ if the co-analytic part of its member is zero. Therefore, the function $f(z)$ ifor the class $\mathcal{SH}$ may be written as (1).

Also, let $\mathcal{NH}$ be the subclass of $\mathcal{SH}$ which consists functions of the form $f=h+\overline{g}$, such that [2],

$$g(z)=-\sum_{n=1}^{\infty}|\beta_n|z^n,$$
$$h(z)=z-\sum_{n=2}^{\infty}|\alpha_n|z^n,|\beta_1|<1 \tag{3}$$

Let $\mathcal{H}(a,b),(a\geq 0,0\leq b<1)$ be the subclass of harmonic functions of the form Eqn. (2) which satisfies [2],

$$\mathcal{R}e\{(h'(z)+g'(z))+3az(h''(z)+g''(z))+az^2(h'''(z)+g'''(z))\}>b. \tag{4}$$

Also, in [2] authors defined, the class $\mathcal{NH}(a,b)$ by $\mathcal{NH}(a,b)=\mathcal{NH}\cap\mathcal{H}(a,b)$.

Hypergeometric functions on Harmonic functions plays an important part in geometric function theory. In 2004, Ahuja and Silverman [3] studied the relationship between distinct hypergeometric functions and harmonic univalent functions. Later, in 2007, Ahuja [4] further investigated by applying certain planar harmonic convolution operators on various subclasses, the connections between the theory of harmonic mappings in the plane and hypergeometric functions. There are many other important studies in this connection [3-11, 12].

In this paper, we study harmonic univalent functions associated with ℓ-Hypergeometric function (in short, ℓ-H function). For $z\in\mathbb{C}$, the ℓ-H function is defined as

$$H\begin{bmatrix}\alpha; & z\\ \beta;\ (\gamma:\ell) & \end{bmatrix}=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n(\gamma)_n^{\ell n}}\frac{z^n}{n!} \tag{5}$$

where $\ell,\alpha\in\mathbb{C}$ with $\mathcal{R}e(\ell)\geq 0$, $\beta,\gamma\in\mathbb{C}\backslash\{0,-1,-2,\cdots\}$ and $(c)_n=\frac{\Gamma(c+n)}{\Gamma(c)}$. If we put $\ell=0$ in (5), then ℓ-H function turns to well known confluent hypergeometric function,

$$H\begin{bmatrix}\alpha; & z\\ \beta;(\gamma:0) & \end{bmatrix}={}_1F_1\begin{bmatrix}\alpha; & z\\ \beta; & \end{bmatrix} \tag{6}$$

The ℓ-H function (5) recently studied by Chudasama and Dave [10].

Next, let $\mathcal{H}(z)=\mathcal{H}_1(z)+\overline{\mathcal{H}_2(z)}$, in which

$$\mathcal{H}_1(z)=zH\begin{bmatrix}\alpha_1; & z\\ \beta_1;(c:\ell_1) & \end{bmatrix}=z+\sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}}{(n-1)!(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}z^n, \tag{7}$$

$$\mathcal{H}_2(z)=H\begin{bmatrix}\alpha_2; & z\\ \beta_2;(c:\ell_2) & \end{bmatrix}-1=\sum_{n=1}^{\infty}\frac{(\alpha_2)_n}{n!(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}z^n \tag{8}$$

We organize the paper in the following way. In Section II, we provide some definitions and lemmas which are useful in our main results. In section III, we derive the necessary and sufficient conditions for harmonic functions connected with ℓ-H functions to be in the classes $\mathcal{H}(a,b)$ and in $\mathcal{NH}(a,b)$.

II. PRELIMINARIES

In this section we present few definitions and lemmas which are useful in the sequel.

Lemma 1. [10] The ℓ-Hypergeometric function is an entire function of z, provided $\mathcal{R}e(\ell)\geq 0$ and $\mathcal{R}e\left(c\,\ell-\frac{\ell}{2}+1\right)>0$.

Definition 2. [10] Let $f(z)=\sum_{n=1}^{\infty}a_n z^n,\quad z\in\mathbb{C}\backslash\{0\}$, $k\in\mathbb{N}\cup\{0\}$ and $p\in\mathbb{C}$. Define

$$ {}_k\Delta_p^{\mathcal{D}}f(z)=\begin{cases}\sum_{n=1}^{\infty}a_n(p)_{n-1}^{k}(\mathcal{D}+p-1)^{kn}z^n, k\in\mathbb{N},\\ f(z), k=0,\end{cases}$$

where $\mathcal{D}$ is the Euler differential operator given by $\mathfrak{D}=z\frac{d}{dz}$.

From the above definition it can be seen that the ℓ-H function (5) satisfies the differential equation

$$({}_\ell\Delta_c^{\mathcal{D}})(\mathcal{D}+b-1)\mathcal{D}w-z(\mathcal{D}+a)w=0 \quad (9)$$

for $z,a\in\mathbb{C}$ and $c,b\in\mathbb{C}\backslash\{0,-1,-2,\cdots\}$ and $\ell=0,1,2,\cdots$. It was established in [7].

Definition 3. [10] The ℓ-H exponential function is defined as

$$e_H^{\ell}(z)=H\left[\begin{matrix}-; & z\\ -;(1:\ell)\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{z^n}{(n!)^{\ell n+1}}, \quad (10)$$

for all $z\in\mathbb{C}$ and $\mathcal{R}e(\ell)\geq 0$.

Lemma 4. [2] Let h and g be given by (2) such that $f=h+\overline{g}$ and the following condition holds:

$$\sum_{n=1}^{\infty}[n+an(n^2-1)](|\alpha_n|+|\beta_n|)+b\leq 1, \quad (11)$$

where $0\leq b<1$, $\alpha_1=1, a\geq 0$ then f is harmonic univalent, sense-preserving in $\mathbb{D}_1(0)$ and $f\in\mathcal{H}(a,b)$.

Lemma 5. [2] Let $f=h+\overline{g}$ be such that h iand g are given iby (2). Then $f\in\mathcal{NH}(a,b)$ if and only if

$$\sum_{n=2}^{\infty} n[1+a(n^2-1)]|\alpha_n| +\sum_{n=1}^{\infty}[n+a(n^2-1)n]|\beta_n|+b\leq 1, \quad (12)$$

in which $0\leq b<1,\ \alpha_1=1, a\geq 0$.

III. MAIN RESULTS

Theorem 6. If $a_i\leq b_i, c_i\geq 4$ and $\ell_i\geq 1$, for $i=1,2$. Then the sufficient conditions for $\mathcal{H}=\mathcal{H}_1+\overline{\mathcal{H}_2}$ to be harmonic univalent and sense-preserving in $\mathbb{D}_1(0)$ and $\mathcal{H}(z)\in\mathcal{H}(a,b)$ are that

$$(\gamma_1^2-3\gamma_1-1)-\gamma_1^2(\gamma_2+1)\geq 0 \quad (13)$$

$$\begin{cases} 1+\frac{2}{\gamma_1}+a[\frac{6}{\gamma_1}+\frac{27}{\gamma_1(\gamma_1+1)}\\ +\frac{256}{\gamma_1(\gamma_1+1)(\gamma_1+2)}]+[\frac{1+3a}{\gamma_1}]\left[\frac{1+\gamma_1}{\gamma_1}\right]\\ +[\frac{1+a}{\gamma_2}][\frac{1+\gamma_2}{\gamma_2}]-a[\frac{3}{\gamma_1}+\frac{1}{\gamma_2}]\leq 2-b,\end{cases} \quad (14)$$

where $a\geq 0$ and $0\leq b<1$.

Proof. Let $\mathcal{H}(z)=\mathcal{H}_1(z)+\overline{\mathcal{H}_2(z)}$. Then

$$\mathcal{H}(z)=z+\sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{z^n}{(n-1)!} +\sum_{n=1}^{\infty}\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{z^n}{n!}.$$

Firstly, we prove $\mathcal{H}$ is locally univalent and sense-preserving in $\mathbb{D}_1(0)$. For this, consider

$$\begin{aligned}|\mathcal{H}_1'(z)| &=\left|1+\sum_{n=2}^{\infty}\frac{n(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{z^{n-1}}{(n-1)!}\right|\\ &>1-\sum_{n=2}^{\infty}\frac{n(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{|z|^{n-1}}{(n-1)!}\\ &\geq 1-\sum_{n=2}^{\infty}\frac{n}{(\gamma_1)_{n-1}^{\ell_1(n-1)}}.\end{aligned}$$

Since $\gamma_1\geq 4>1$ and $\ell_1\geq 1$, it follows that

$(\gamma_1)_n\leq(\gamma_1)_n^{\ell_1 n}$,

for all $n\in\mathbb{N}$ and

$$\begin{aligned}\frac{n}{(c)_{n-1}} &=\frac{n}{c(c+1)(c+2)\cdots(c+n-2)}\\ &<\frac{1}{c(c+1)^{n-3}}\end{aligned}$$

for $n\in\mathbb{N}\backslash\{1,2\}$

So,

$$\begin{aligned}|\mathcal{H}'_1(z)| &\geq 1-\sum_{n=2}^{\infty}\frac{n}{(\gamma_1)_{n-1}}\\ &=1-\frac{2}{\gamma_1}-\frac{1}{\gamma_1}\sum_{n=0}^{\infty}\frac{1}{(\gamma_1+1)^n}\\ &=1-\frac{1+3\gamma_1}{\gamma_1^2}\\ &=\frac{\gamma_1^2-3\gamma_1-1}{\gamma_1^2}.\end{aligned}$$

On the other hand, by similar arguments, we have

$$\begin{aligned}|\mathcal{H}'_2(z)| &=\sum_{n=1}^{\infty}\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{|z|^{n-1}}{(n-1)!}\\ &\leq\sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n^{\ell_2 n}}\\ &\leq\sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n}\\ &=\sum_{n=1}^{\infty}\frac{1}{\gamma_2(\gamma_2+1)(\gamma_2+2)\cdots(\gamma_2+n-1)}\\ &<\frac{1}{\gamma_2}\sum_{n=0}^{\infty}\frac{1}{(\gamma_2+1)^n}=\frac{\gamma_2+1}{\gamma_2^2}.\end{aligned}$$

From the condition (13), w have

$|\mathcal{H}'_1(z)|>|\mathcal{H}'_2(z)|$.

Now to prove $\mathcal{H}$ is univalent in $\mathbb{D}_1(0)$, we suppose $z_1,z_2\in\mathbb{D}_1(0)$ such that $z_1\neq z_2$. As $\mathbb{D}_1(0)$ is convex and simply connected, we have $z(t)=(1-t)z_1+tz_2\in\mathbb{D}_1(0)$, where $0\leq t\leq 1$. Therefore, we have

$$\mathcal{H}(z_1)-\mathcal{H}(z_2)=\int_0^1[(z_2-z_1)\mathcal{H}_1'(z(t)) +\overline{(z_2-z_1)\mathcal{H}_2'(z(t))}]dt$$

such that

$$\begin{aligned}&\mathcal{R}e\left\{\frac{\mathcal{H}(z_1)-\mathcal{H}(z_2)}{z_2-z_1}\right\}\\ &=\int_0^1\mathcal{R}e\left[\mathcal{H}_1'(z(t))+\frac{\overline{(z_2-z_1)}}{(z_2-z_1)}\overline{\mathcal{H}_2'(z(t))}\right]dt \quad (15)\\ &>\int_0^1[\mathcal{R}e\,(\mathcal{H}_1'(z(t)))+|\mathcal{H}_2'(z(t))|]dt,\end{aligned}$$

where

$$\begin{aligned}&\mathcal{R}e\,(\mathcal{H}_1'(z(t)))+|\mathcal{H}_2'(z(t))|\\ &\geq 1-\sum_{n=2}^{\infty}n\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{|z|^{n-1}}{(n-1)!}\\ &-\sum_{n=1}^{\infty}\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{|z|^{n-1}}{(n-1)!}\\ &\geq 1-\sum_{n=2}^{\infty}\frac{n}{(\gamma_1)_{n-1}^{\ell_1(n-1)}}-\sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n^{\ell_2 n}}\\ &\geq 1-\sum_{n=2}^{\infty}\frac{n}{(\gamma_1)_{n-1}}-\sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n}\\ &\geq 1-\frac{2}{\gamma_1}-\frac{1}{\gamma_1}\sum_{n=0}^{\infty}\frac{1}{(\gamma_1+1)^n}-\frac{1}{\gamma_2}\sum_{n=0}^{\infty}\frac{1}{(\gamma_2+1)^n}\\ &=\frac{\gamma_1^2-3\gamma_1-1}{\gamma_1^2}-\frac{\gamma_2+1}{\gamma_2^2}\geq 0.\end{aligned}$$

Thus, by (15), we get $\mathcal{H}(z_1)\neq\mathcal{H}(z_2)$ and hence $\mathcal{H}$ is univalent in $\mathcal{D}_1(0)$. Finally, we prove $\mathcal{H}\in\mathcal{H}(a,b)$.

In order to prove this, it is suffices to prove condition (11). i.e.,

$$\sum_{n=1}^{\infty} n[1+ \quad a(n^2-1)][\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{1}{(n-1)!} + \frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{1}{n!}] \leq 2-b. \qquad (16)$$

Under the given conditions, we have

$$\frac{n^3}{(\gamma_1)_{n-1}} \leq \frac{n^2}{\gamma_1(\gamma_1+1)_{n-3}} = \frac{n(n-3)}{\gamma_1(\gamma_1+1)_{n-3}} + \frac{3n}{\gamma_1(\gamma_1+1)_{n-3}} \leq \frac{1}{\gamma_1(\gamma_1+1)^{n-5}} + \frac{3}{\gamma_1(\gamma_1+1)^{n-4}}$$

for $n \in \mathbb{N}\backslash\{1,2,3,4\}$,

$$\frac{n^2}{(\gamma_2)_n} = \frac{n(n-1)}{(\gamma_2)_n} + \frac{n}{(\gamma_2)_n} \leq \frac{1}{\gamma_2(\gamma_2+1)^{n-3}} + \frac{1}{\gamma_2(\gamma_2+1)^{n-2}}$$

for $n \in \mathbb{N}\backslash\{1,2\}$,

$$\frac{n}{(\gamma_1)_{n-1}} \leq \frac{1}{\gamma_1(\gamma_1+1)^{n-3}} \text{ for } n \in \mathbb{N}\backslash\{1,2\},$$

and $\frac{1}{(\gamma_2)_n} \leq \frac{1}{\gamma_2(\gamma_2+1)^{n-1}}$ for $n \in \mathbb{N}$.

Now, consider

$$\begin{aligned}
&\sum_{n=1}^{\infty} n[1+a(n^2-1)][\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{1}{(n-1)!} + \frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{1}{n!}] \\
&= \sum_{n=1}^{\infty} (1-a)n\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{1}{(n-1)!} + \sum_{n=1}^{\infty} an^3\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{1}{(n-1)!} \\
&+ \sum_{n=1}^{\infty} (1-a)n\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{1}{n!} + \sum_{n=1}^{\infty} an^3\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{1}{n!} \\
&\leq (1-a)\sum_{n=1}^{\infty}\frac{n}{(\gamma_1)_{n-1}} + a\sum_{n=1}^{\infty}\frac{n^3}{(\gamma_1)_{n-1}} + (1-a)\sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n} + a\sum_{n=1}^{\infty}\frac{n^2}{(\gamma_2)_n} \\
&= (1-a)\left[1+\frac{2}{\gamma_1}\right] + (1-a)\sum_{n=3}^{\infty}\frac{n}{(\gamma_1)_{n-1}} + a\left[1+\frac{8}{\gamma_1}+\frac{27}{\gamma_1(\gamma_1+1)}+\frac{256}{\gamma_1(\gamma_1+1)(\gamma_1+2)}\right] \\
&+ a\sum_{n=5}^{\infty}\frac{n^3}{(\gamma_1)_{n-1}} + (1-a)\sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n} + a\left[\frac{1}{\gamma_2}+\frac{4}{\gamma_2(\gamma_2+1)}\right] + a\sum_{n=3}^{\infty}\frac{n^2}{(\gamma_2)_n} \\
&= 1+\frac{2}{\gamma_1} + a\left[\frac{6}{\gamma_1}+\frac{27}{\gamma_1(\gamma_1+1)}+\frac{256}{\gamma_1(\gamma_1+1)(\gamma_1+2)}\right] + \left[\frac{1+3a}{\gamma_1}\right]\sum_{n=0}^{\infty}\frac{1}{(\gamma_1+1)^n} \\
&+ \left[\frac{1+a}{\gamma_2}\right]\sum_{n=0}^{\infty}\frac{1}{(\gamma_2+1)^n} - a\left[\frac{3}{\gamma_1}+\frac{1}{\gamma_2}\right] \\
&= 1+\frac{2}{\gamma_1} + a\left[\frac{6}{\gamma_1}+\frac{27}{\gamma_1(\gamma_1+1)}+\frac{256}{\gamma_1(\gamma_1+1)(\gamma_1+2)}\right] + \left[\frac{1+3a}{\gamma_1}\right]\left[\frac{1+\gamma_1}{\gamma_1}\right] \\
&+ \left[\frac{1+a}{\gamma_2}\right]\left[\frac{1+\gamma_2}{\gamma_2}\right] - a\left[\frac{3}{\gamma_1}+\frac{1}{\gamma_2}\right] \\
&\leq 2-b.
\end{aligned}$$

This completes the proof.

Define a function,

$$\mathcal{F}_1(z) = 2z - \mathcal{H}_1(z) - \overline{\mathcal{H}_2(z)} = z - \sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{z^n}{(n-1)!} - \overline{\sum_{n=1}^{\infty}\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{z^n}{n!}}$$

Theorem 7. Let $a \geq 0$, $0 \leq b < 1$, $a_i \leq b_i, c_i \geq 4$ and $\ell_i \geq 1$, for $i = 1,2$ and $\alpha_2 < \beta_2\gamma_2^{\ell_2}$. Then $\mathcal{F}_1(z) \in \mathcal{NH}(a,b)$ if and only if (13) and (14) holds.

Proof. By definition, it is clear that $\mathcal{F}_1(z) \in \mathcal{NH}$. Now suppose Eqns. (13) and (14) holds. Then, by Theorem 3 $\mathcal{F}_1(z) \in \mathcal{H}(a,b)$.

Hence, $\mathcal{F}_1(z) \in \mathcal{NH}(a,b)$.

Conversely, suppose

$\mathcal{F}_1(z) \in \mathcal{NH}(a,b)$.

Since $\mathcal{NH}(a,b) \subset \mathcal{H}(a,b)$, $\mathcal{F}_1(z) \in \mathcal{H}(a,b)$. So, $\mathcal{F}_1$ satisfies the inequalities (13) and (16) by Theorem 3 and hence (14) holds.

Theorem 8. Let $a \geq 0$, $0 \leq b < 1$, $a_i \leq b_i, c_i \geq 4$ and $\ell_i \geq 1$, for $i = 1,2$ and $\alpha_2 < \beta_2\gamma_2^{\ell_2}$. Then the necessary and sufficient condition for $f \star (\mathcal{H}_1 + \overline{\mathcal{H}_2}) \in \mathcal{NH}(a,b)$, where $f \in \mathcal{NH}(a,b)$ is that

$$\frac{\gamma_1+1}{\gamma_1^2} + \frac{\gamma_2+1}{\gamma_2^2} \leq 1.$$

Proof. Let $f = h + \overline{g} \in \mathcal{NH}(a,b)$, in which h and g defined by (2).

Then

$$f \star (\mathcal{H}_1 + \overline{\mathcal{H}_2})(z) = h(z) \star \mathcal{H}_1(z) + \overline{g(z) \star \mathcal{H}_2(z)} = z - \sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{z^n}{(n-1)!} - \overline{\sum_{n=1}^{\infty}\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{z^n}{n!}}.$$

By Lemma 2,

$f \star (\mathcal{H}_1 + \overline{\mathcal{H}_2}) \in \mathcal{NH}(a,b)$ iff

$$\sum_{n=1}^{\infty}[n+a(n^2-1)n] \quad [\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{a_n}{(n-1)!} + \frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{b_n}{n!}] \leq 2-b.$$

By Lemma 2,

$$\sum_{n=1}^{\infty}[n+a(n^2-1)n](a_n+b_n) + b \leq 2.$$

That is,

$$\sum_{n=2}^{\infty} n[1+a(n^2-1)]a_n + \sum_{n=1}^{\infty} n[1+a(n^2-1)]b_n \leq 1-b,$$

$n[1+a(n^2-1)]a_n \leq 1-b$ and

$n[1+a(n^2-1)]b_n \leq 1-b$

$$a_n \leq \frac{1-b}{n[1+a(n^2-1)]} \text{ and}$$

$$b_n \leq \frac{1-b}{n[1+a(n^2-1)]}, (n \geq 1) \qquad (17)$$

Now, from Eqn. (17), we have

$$\sum_{n=1}^{\infty} n[\quad 1+a(n^2-1)][\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{a_n}{(n-1)!} + \frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{b_n}{n!}]$$

$$\leq \sum_{n=2}^{\infty}(1-b)\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{1}{(n-1)!} + \sum_{n=1}^{\infty}(1-b)\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{1}{n!}$$

$$= (1-b)[\sum_{n=1}^{\infty}\frac{(\alpha_1)_n}{(\beta_1)_n(\gamma_1)_n^{\ell_1 n}}\frac{1}{n!} + \sum_{n=1}^{\infty}\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{1}{n!}]$$

$$\leq (1-b)\left[\sum_{n=1}^{\infty}\frac{1}{(\gamma_1)_n^{\ell_1 n}} + \sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n^{\ell_2 n}}\right]$$

$$\leq (1-b)\left[\sum_{n=1}^{\infty}\frac{1}{(\gamma_1)_n} + \sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n}\right]$$

$$\leq (1-b)[\sum_{n=1}^{\infty}\frac{1}{\gamma_1(\gamma_1+1)^{n-1}} + \sum_{n=1}^{\infty}\frac{1}{\gamma_2(\gamma_2+1)^{n-1}}]$$

$$\leq (1-b)\left[\frac{\gamma_1+1}{\gamma_1^2}+\frac{\gamma_2+1}{\gamma_2^2}\right]$$

This completes the proof.

Theorem 9. Let $a_i \leq b_i, c_i \geq 4$ and $\ell_i \geq 1$, for $i = 1,2$ and $\alpha_2 < \beta_2\gamma_2^{\ell_2}$. Then the necessary and sufficient condition for a function

$$\mathcal{F}_2(z) = \int_0^z H\left[\begin{matrix}\alpha_1; \\ \beta_1;(\gamma_1:\ell_1)\end{matrix}\; t\right]dt + \int_0^z \left[H\left[\begin{matrix}\alpha_2; \\ \beta_2;(\gamma_2:\ell_2)\end{matrix}\; t\right]-1\right]dt$$

to be in $\mathcal{H}(a,b)$ is that

$$[1+3a][\frac{\gamma_1+1}{\gamma_1^2} \quad + \frac{\gamma_2+1}{\gamma_2^2}] + a\left[\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right] \leq 1-b,$$

where $a \geq 0$ and $0 \leq b < 1$.

Proof. From Lemma 2, the function

$$\mathcal{F}_2(z) = z \quad + \sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}}\frac{z^n}{(n-1)!\,n} + \sum_{n=2}^{\infty}\frac{(\alpha_2)_{n-1}}{(\beta_2)_{n-1}(\gamma_2)_{n-1}^{\ell_2(n-1)}}\frac{z^n}{(n-1)!\,n}$$

is in $\mathcal{H}(a,b)$ if

$$\sum_{n=2}^{\infty} n[1+ \quad a(n^2-1)][\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}(n-1)!\,n} + \frac{(\alpha_2)_{n-1}}{(\beta_2)_{n-1}(\gamma_2)_{n-1}^{\ell_2(n-1)}(n-1)!\,n}] \leq 1-b.$$

So, consider

$$\sum_{n=2}^{\infty}[1+a(n^2-1)][\frac{(\alpha_1)_{n-1}}{(\beta_1)_{n-1}(\gamma_1)_{n-1}^{\ell_1(n-1)}(n-1)!} + \frac{(\alpha_2)_{n-1}}{(\beta_2)_{n-1}(\gamma_2)_{n-1}^{\ell_2(n-1)}(n-1)!}]$$

$$= \sum_{n=1}^{\infty}\frac{(\alpha_1)_n}{(\beta_1)_n(\gamma_1)_n^{\ell_1 n}n!} + \sum_{n=1}^{\infty}\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}n!}$$

$$+a\sum_{n=1}^{\infty}n^2\frac{(\alpha_1)_n}{(\beta_1)_n(\gamma_1)_n^{\ell_1 n}n!} + a\sum_{n=1}^{\infty}n^2\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}n!}$$

$$+2a\sum_{n=1}^{\infty}n\frac{(\alpha_1)_n}{(\beta_1)_n(\gamma_1)_n^{\ell_1 n}n!} + 2a\sum_{n=1}^{\infty}n\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}n!}$$

$$\leq \sum_{n=1}^{\infty}\frac{1}{(\gamma_1)_n} + \sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n} + a\sum_{n=1}^{\infty}\frac{n}{(\gamma_1)_n} + a\sum_{n=1}^{\infty}\frac{n}{(\gamma_2)_n}$$

$$+2a\sum_{n=1}^{\infty}\frac{1}{(\gamma_1)_n} + 2a\sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n}$$

$$= (1+2a)\sum_{n=1}^{\infty}\frac{1}{(\gamma_1)_n} + (1+2a)\sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n}$$

$$+a\sum_{n=1}^{\infty}\frac{n}{(\gamma_1)_n} + a\sum_{n=1}^{\infty}\frac{n}{(\gamma_2)_n}$$

$$\leq [1+2a]\left[\frac{\gamma_1+1}{\gamma_1^2}+\frac{\gamma_2+1}{\gamma_2^2}\right] + a\left[\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right]$$

$$+a[\sum_{n=2}^{\infty}\frac{1}{\gamma_1(\gamma_1+1)^{n-2}} + \sum_{n=2}^{\infty}\frac{1}{\gamma_2(\gamma_2+1)^{n-2}}]$$

$$\leq [1+2a]\left[\frac{\gamma_1+1}{\gamma_1^2}+\frac{\gamma_2+1}{\gamma_2^2}\right] + a\left[\frac{2\gamma_1+1}{\gamma_1^2}+\frac{2\gamma_2+1}{\gamma_2^2}\right]$$

$$\leq [1+3a]\left[\frac{\gamma_1+1}{\gamma_1^2}+\frac{\gamma_2+1}{\gamma_2^2}\right] + a\left[\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right]$$

$$\leq 1-b.$$

Theorem 9. Let $a_i \leq b_i, c_i \geq 4$ and $\ell_i \geq 1$, for $i = 1,2$ and $\alpha_2 < \beta_2\gamma_2^{\ell_2}$. Then the necessary and sufficient condition for

$$\mathcal{F}_3(z) = \int_0^z H\left[\begin{matrix}\alpha_1; \\ \beta_1;(\gamma_1:\ell_1)\end{matrix}\; t\right]dt - \int_0^z \left[H\left[\begin{matrix}\alpha_2; \\ \beta_2;(\gamma_2:\ell_2)\end{matrix}\; t\right]-1\right]dt$$

to be in $\mathcal{H}(a,b)$ is that

$$\left[\frac{4a+\gamma_1+1}{\gamma_1^{1+\ell_1}}\right] + (1+4a) \quad \left[\frac{1+\gamma_2}{\gamma_2^2}\right] + \frac{a}{\gamma_2} \leq 1-b,$$

where $a \geq 0$ and $0 \leq b < 1$.

Proof. From Lemma 5, the function

$$\mathcal{F}_3(z) = z - [\frac{|\alpha_1|}{\beta_1\gamma_1^{\ell_1}}\sum_{n=2}^{\infty}\frac{(\alpha_1+1)_{n-2}}{(\beta_1+1)_{n-2}(\gamma_1+1)_{n-2}^{\ell_1(n-2)}} \times \frac{z^n}{(n-1)!\,n}] - \sum_{n=2}^{\infty}\frac{(\alpha_2)_{n-1}}{(\beta_2)_{n-1}(\gamma_2)_{n-1}^{\ell_2(n-1)}}\frac{z^n}{(n-1)!\,n}$$

is in $\mathcal{H}(a,b)$ if

$$\sum_{n=2}^{\infty} n[1+a(n^2-1)] \times [\frac{|\alpha_1|}{\beta_1\gamma_1^{\ell_1}}\frac{(\alpha_1+1)_{n-2}}{(\beta_1+1)_{n-2}(\gamma_1+1)_{n-2}^{\ell_1(n-2)}}\frac{1}{(n-1)!\,n} + \frac{(\alpha_2)_{n-1}}{(\beta_2)_{n-1}(\gamma_2)_{n-1}^{\ell_2(n-1)}}\frac{1}{(n-1)!\,n}] \le 1-b.$$

So, consider

$$\sum_{n=2}^{\infty} [1+a(n^2-1)] \times [\frac{|\alpha_1|}{\beta_1\gamma_1^{\ell_1}}\frac{(\alpha_1+1)_{n-2}}{(\beta_1+1)_{n-2}(\gamma_1+1)_{n-2}^{\ell_1(n-2)}}\frac{1}{(n-1)!} + \frac{(\alpha_2)_{n-1}}{(\beta_2)_{n-1}(\gamma_2)_{n-1}^{\ell_2(n-1)}}\frac{1}{(n-1)!}]$$

$$= \sum_{n=2}^{\infty} \frac{|\alpha_1|}{\beta_1\gamma_1^{\ell_1}}\frac{(\alpha_1+1)_{n-2}}{(\beta_1+1)_{n-2}(\gamma_1+1)_{n-2}^{\ell_1(n-2)}}\frac{1}{(n-1)!} + \sum_{n=2}^{\infty}\frac{(\alpha_2)_{n-1}}{(\beta_2)_{n-1}(\gamma_2)_{n-1}^{\ell_2(n-1)}}\frac{1}{(n-1)!}$$

$$+a\sum_{n=2}^{\infty} \frac{|\alpha_1|}{\beta_1\gamma_1^{\ell_1}}\frac{(n^2-1)(\alpha_1+1)_{n-2}}{(\beta_1+1)_{n-2}(\gamma_1+1)_{n-2}^{\ell_1(n-2)}}\frac{1}{(n-1)!} + a\sum_{n=2}^{\infty}(n^2-1)\frac{(\alpha_2)_{n-1}}{(\beta_2)_{n-1}(\gamma_2)_{n-1}^{\ell_2(n-1)}}\frac{1}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{|\alpha_1|}{\beta_1\gamma_1^{\ell_1}}\frac{(\alpha_1+1)_{n}}{(\beta_1+1)_{n}(\gamma_1+1)_{n}^{\ell_1 n}}\frac{1}{(n+1)!} + a\sum_{n=0}^{\infty} n\frac{|\alpha_1|}{\beta_1\gamma_1^{\ell_1}}\frac{(\alpha_1+1)_{n}}{(\beta_1+1)_{n}(\gamma_1+1)_{n}^{\ell_1 n}}\frac{1}{n!}$$

$$+3a\sum_{n=0}^{\infty} \frac{|\alpha_1|}{\beta_1\gamma_1^{\ell_1}}\frac{(\alpha_1+1)_{n}}{(\beta_1+1)_{n}(\gamma_1+1)_{n}^{\ell_1 n}}\frac{1}{n!} + a\sum_{n=1}^{\infty} n^2\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{1}{n!}$$

$$+2a\sum_{n=1}^{\infty} n\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{1}{n!} + (1+a)\sum_{n=1}^{\infty}\frac{(\alpha_2)_n}{(\beta_2)_n(\gamma_2)_n^{\ell_2 n}}\frac{1}{n!}$$

$$\le \frac{1}{\gamma_1^{\ell_1}}\sum_{n=0}^{\infty}\frac{1}{(\gamma_1+1)_n^{\ell_1 n}} + \frac{a}{\gamma_1^{\ell_1}}\sum_{n=1}^{\infty}\frac{1}{(\gamma_1+1)_n^{\ell_1 n}}$$

$$+\frac{3a}{\gamma_1^{\ell_1}}\sum_{n=1}^{\infty}\frac{1}{(\gamma_1+1)_n^{\ell_1 n}}$$

$$+a\sum_{n=1}^{\infty}\frac{n}{(\gamma_2)_n^{\ell_2 n}} + 2a\sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n^{\ell_2 n}}$$

$$+(1+a)\sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n^{\ell_2 n}}$$

$$\le \frac{1}{\gamma_1^{\ell_1}}\left[1+\sum_{n=1}^{\infty}\frac{1}{(\gamma_1+1)_n}\right] + \frac{a}{\gamma_1^{\ell_1}}\sum_{n=1}^{\infty}\frac{1}{(\gamma_1+1)_n}$$

$$+\frac{3a}{\gamma_1^{\ell_1}}\sum_{n=1}^{\infty}\frac{1}{(\gamma_1+1)_n} + a\sum_{n=1}^{\infty}\frac{n}{(\gamma_2)_n} + 2a\sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n}$$

$$+(1+a)\sum_{n=1}^{\infty}\frac{1}{(\gamma_2)_n}$$

$$\le \frac{1}{\gamma_1^{\ell_1}}\left[1+\sum_{n=1}^{\infty}\frac{1}{(\gamma_1+1)^n}\right] + \frac{a}{\gamma_1^{\ell_1}}\sum_{n=1}^{\infty}\frac{1}{(\gamma_1+1)^n}$$

$$+\frac{3a}{\gamma_1^{\ell_1}}\sum_{n=1}^{\infty}\frac{1}{(\gamma_1+1)^n} + a\left[\frac{1}{\gamma_2}+\sum_{n=2}^{\infty}\frac{1}{\gamma_2(\gamma_2+1)^{n-2}}\right]$$

$$+(1+3a)\sum_{n=1}^{\infty}\frac{1}{\gamma_2(\gamma_2+1)^{n-1}}$$

$$\le \frac{1+\gamma_1}{\gamma_1^{1+\ell_1}} + \frac{a}{\gamma_1^{1+\ell_1}} + \frac{3a}{\gamma_1^{1+\ell_1}} + a\left[\frac{1}{\gamma_2}+\frac{1+\gamma_2}{\gamma_2^2}\right]$$

$$+(1+3a)\left[\frac{1+\gamma_2}{\gamma_2^2}\right]$$

$$\le \left[\frac{4a+\gamma_1+1}{\gamma_1^{1+\ell_1}}\right] + (1+4a)\left[\frac{1+\gamma_2}{\gamma_2^2}\right] + \frac{a}{\gamma_2}$$

$$\le 1-b$$

This completes the proof.

Conflict of Interest. The authors declare no conflict of interest associated with this work.

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